Arithmetic Density and Related Concepts

J. R. Wheat

06/18/2023

1 Counting Arithmetic Progressions (APs)

Given an arbitrary increasing sequence of positive integers S there is associated with it, a counting function $\alpha_S(n)$ which determines the number of arithmetic progressions[1] up to $s_n \in S$. The counting function for the positive integers, and consequently all infinite arithmetic progressions, is given by the formula listed under A330285 (OEIS)[2]:

$$\sum_{k=1}^{n}\sum_{j=1}^{k}\left\lfloor \frac{k-1}{j+1}\right\rfloor.$$

We may further generalize from this AP counting function, $\alpha_{\mathbb{Z}^+}$, the following formula which applies to any arbitrary integer sequence,

$$\sum_{k=1}^{n}\alpha_{\mathbb{Z}^+}(k)\pi(n,k)$$

where $\pi(n,k)$ is the number of *primitive progressions* of length k, up to the n-th element of S. A primitive progression is defined as any subset

$$\{s, s+d,\ \dots\ , s+(k-1)d\} \subset \{s_1, s_2,\ \dots\ , s_n\}$$

such that

$$\{s-d, s+kd\} \cap \{s_1, s_2,\ \dots\ , s_n\} = \emptyset,$$

$$\{s, s+d,\ \dots\ , s+(k-1)d\} \not\subset \{s', s'+d',\ \dots\ , s'+(j-1)d'\}$$

for $k-1 \leq j$ and $d' < d$.

2 Relative Arithmetic Density

There are two other uses for $\alpha_{\mathbb{Z}^+}$ which will be discussed in both this section and the following section, respectively. The first such use involves computing the *relative arithmetic density* of our sequence S, but before proceeding it is necessary that we elaborate on the formulation of "partial" densities.

Let us begin by observing that the number of APs up to s_n is merely a fraction of all sub-sequences over said interval. The partial density $D_S(n)$ is thus given by

$$\frac{\alpha_S(n)}{2^n - T_n - 1}$$

where T_n is the n-th triangular number and the denominator counts sub-sequences with more than two elements. The infinite sum of partial densities over $\mathbb{Z}^+$ is approximately 2.89563562435821120303, and will be referred to as Layman's constant[3]. The exact value of the infinite series of partial densities requires further analysis, in the meantime we can still use Layman's constant to construct a definition for relative densities in general by comparing arbitrary number sequences with the positive integers themselves. The relative arithmetic density is thus described by the ratio:

$$\mathcal{D}(S) = \frac{\sum_{n=1}^{\infty} D_S(n)}{\sum_{n=1}^{\infty} D_{\mathbb{Z}^+}(n)},$$

which is just some real number less than or equal to one.

3 Natural Arithmetic Density

The second use for these counting functions involves a more naturalistic interpretation of arithmetic density, as far as real number valuations are concerned. We can arrive at such a definition by taking advantage of the fact that the counting function for any infinite progression has maximal growth. It then becomes an arbitrary choice of which complementary sequence we might use to construct a suitable infinite series for our valuation. The author has chosen the sequence of factorials because firstly they converge for all AP counting functions, and secondly because they allow for more convenient algebraic analysis. The formula for the *natural arithmetic density* of S is then defined to be

$$\delta(S) = \sum_{n=1}^{\infty} \frac{\alpha_S(n)}{(n-1)!}.$$

This series is approximately 1.42638919894259387857 when the counting function is taken over any infinite AP. For the rare case when there is only one progression of length three at the beginning of the given sequence, the series above is equal to $e - 2$.

4 The Partitive Convergence Theorem

4.1 Proof of the Theorem

Assuming that the infinite series

$$\sum_{n=1}^{\infty} \frac{1}{s_n} = \infty,$$

then there exists some sub-sequence $\{s'_1, s'_2, \ ... \ \}$ such that

$$\sum_{k=n_1}^{n_1} \frac{1}{s_k} < \sum_{k=n_1+1}^{n_2} \frac{1}{s_k} < \sum_{k=n_2+1}^{n_3} \frac{1}{s_k} < ...$$

and

$$\sum_{k=n_1}^{n_1} \frac{1}{s_k} > \sum_{k=n_1+1}^{n_2-1} \frac{1}{s_k} > \sum_{k=n_2+1}^{n_3-1} \frac{1}{s_k} > ... \ ,$$

for $n_1 = 1$ and $s'_m = s_{n_m}$. We can deduce that this sequence of finite series must be constructable, otherwise the infinite sum of reciprocals would not diverge. What is not immediately evident, however, is that each of these finite series should converge to the same value from below. We denote this *divergence constant* as C and we will further expand on its significance in the following subsection.

In order to establish our proof, let us first draw our attention to the terminal values at the end of each finite series. Without the addition of these single terms, any particular series falls short of those before it according to the second inequality above. Therefore, if all of the finite series are to diverge from any particular constant, that would imply each terminal value is greater than the previous one, or $\frac{1}{s'_m} < \frac{1}{s_{m+1}}'$. This cannot be the case since our infinite sub-sequence is monotonically increasing, and so its reciprocal elements must tend toward zero. As an example of the proof, let it be noted that the divergence constant for the harmonic series is log(3).

Before leaving the present topic, there are two other constants which should be briefly discussed for future reference. These are the *criticality constant* κ and the *co-criticality constant* $\bar{\kappa}$. These can be calculated using the infinite series

$$\sum_{m=1}^{\infty} \frac{1}{n_m}$$

and

$$\sum_{m=1}^{\infty} \frac{1}{s'_m} \ ,$$

respectively. In the case where $S = \mathbb{Z}^+$ these two constants are equal, and their value is

$$\sum_{m=1}^{\infty} \frac{2}{3^m - 1} = 2\left(\frac{\log(\frac{3}{2}) - \Psi_{\frac{1}{3}}(1)}{\log(3)}\right)$$

where Ψ_q is the q-digamma function[4].

4.2 Algebraic Derivations

A further consequence of the theorem is that partial sums of divergent series can now be represented using the divergence constant mentioned above. To do so requires the introduction of an error function which adjusts the constant according to the interval over which the function is taken, i.e

$$\sum_{k=n_{m-1}+1}^{n_m} \frac{1}{s_k} = C - \mathrm{Err}(m) \ .$$

According to this definition, it follows that the partial sum up to s'_m is given by the formula

$$\sum_{k=1}^{n_m} \frac{1}{s_k} = Cm - \sum_{k=1}^{m} \mathrm{Err}(k)$$

We can then expand upon this equation by introducing *balancing coefficients* $\beta_{m,k}$ for $n_{m-1}+1 \leq k \leq n_m$, which allow us to compute partial sums over the m-th interval $\{s_{n_{m-1}}, s_{n_{m-1}+1}, \ \dots \ , s_{n_m}\}$. These are implemented in the following manner

$$C\beta_{m,k} - \mathrm{Err}(m) = \sum_{j=n_{m-1}+1}^{k} \frac{1}{s_j} \ ,$$

which then implies

$$C \sum_{j=1}^{\nabla n_m - 1} \beta_{m,j} - (\nabla n_m - 1)\mathrm{Err}(m) = \sum_{k=n_{m-1}+1}^{n_m} \sum_{j=1}^{\nabla n_m - 1} \frac{\nabla n_m - j}{s_k}$$

where ∇ is the backward difference operator[5].

REFERENCES

[1] Weisstein, Eric W. "Arithmetic Progression." From *Mathworld*–A Wolfram web resource. (http://mathworld.wolfram.com/ArithmeticProgression.html)

[2] "A330285." *The On-Line Encyclopedia of Integer Sequences.* (http://oeis.org/A330285)

[3] "A05133." *The On-Line Encyclopedia of Integer Sequences.* (http://oeis.org/A051336)

[4] Weisstein, Eric W. "q-Polygamma Function." From *Mathworld*–A Wolfram web resource. (https://mathworld.wolfram.com/q-PolygammaFunction)

[5] Weisstein, Eric W. "Backward Difference." From *Mathworld*–A Wolfram web resource. (http://mathworld.wolfram.com/BackwardDifference)